the solution of problem (9), (10); $E_9 = \text{const} > 0.$

If the premises of Theorem 4 are fulfilled, estimate (32) for w yields the following relation for $u(t, x, y)$. We write

$$\Phi_q^*(u/U, t, x) = \int_0^{u/U} \left(\sum_{i=0}^{q} Y_i(x, s)\, t^{n-1/2+i/2}\right)^{-1} ds$$

where $Y_i(\xi, \eta)$ are the solutions of system (30) with conditions (31). Then

$$|y^{-1}\Phi_q^*(u/U, t, x)\, t^n - 1| \leqslant E_{10}\, t^{1/2(q+1)}, \quad E_{10} = \text{const} > 0 \tag{41}$$

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ASYMPTOTIC METHOD IN THE PROBLEM OF OSCILLATIONS OF A STRONGLY VISCOUS FLUID

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In [1] the authors have proved a theorem on the existence of solution of the Cauchy's problem for linearized equations corresponding to the problem of motion about a fixed point of a rigid body, with a cavity partially filled with a viscous incompressible fluid. In the case of small Reynolds numbers (high viscosity fluids), these equations will contain a small parameter $\varepsilon = \nu^{-1}$ and the Krylov-Bogoliubov asymptotic method given in [2] can be used to solve the system of Navier-Stokes equations. In the present paper we derive formulas for the corresponding approximate solutions. The case of a highly viscous fluid filling the cavity completely was investigated by Chernous'ko in [3 and 4].

1. Statement of the problem. We assume that a body with a cavity partially filled with a viscous incompressible fluid performs a given motion about a fixed point with an instantaneous angular velocity $\boldsymbol{\omega}$. It is required to determine the motion of fluid in the vessel. In the linearized formulation this problem reduces to solution of the following system of Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} = -\nabla q + \nu\Delta\mathbf{u}, \qquad \operatorname{div} \mathbf{u} = 0 \tag{1.1}$$

in the region Ω filled with fluid in the state of equilibrium, with the boundary conditions

$$\mathbf{u} = 0 \tag{1.2}$$

given on the part Γ_1 of the boundary of Ω corresponding to the cavity wall,

$$\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0, \quad \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0, \quad \frac{\partial}{\partial t}\left(q - 2\nu \frac{\partial u_z}{\partial z}\right) = g u_z \tag{1.3}$$

given on the free surface Γ_0 of the fluid, and with the initial conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad q|_{t=0} = q_0 \qquad \left(q = \frac{p}{\rho} - gz + C\right) \tag{1.4}$$

Here $\mathbf{u}$ is the vector of relative velocity of the fluid, $\mathbf{r}$ is the radius vector relative to the fixed point, p is the pressure, g is the acceleration due to gravity, ρ is the density, ν is the kinematic coefficient of viscosity and C is a constant.

We naturally assume that at high viscosities, motion of the fluid will consist of three components: a forced motion caused by the forces responsible for the given motion of the body; a rapidly decaying motion connected with the initial distribution of velocities and a slowly decaying motion related to the initial position of the free surface.

The asymptotic method proposed below enables us to split the solution of the considered problem into three parts indicated above.

2. Asymptotic method of solution. We consider the following differential equation in a Banach space:

$$\varepsilon \frac{dx}{dt} = Ax + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k B_k(t) x + \sum_{k=0}^{\infty} \varepsilon^k f_k(t) \tag{2.1}$$

Here A is an infinite generating operator of a contraction semigroup, operators B_k are bounded and functions f_k are given. Since Eq. (2.1) differs somewhat from those discussed in [2], we give a brief derivation of the asymptotic expansions for its solutions.

In deriving these asymptotic expansions in the powers of a small parameter ε we encounter two distinct cases, when the operator A has a bounded inverse and when it has not.

1) Let the operator A^{-1} be bounded. Then solutions of the homogeneous equation (2.1) are rapidly decaying functions of t and we can seek a particular solution of the inhomogeneous equation in the form

$$x(t) = h_0(t) + \varepsilon h_1(t) + \varepsilon^2 h_2(t) + \ldots \tag{2.2}$$

Inserting (2.2) into (2.1) and comparing the coefficients ε of like powers we obtain

$$h_0 = A^{-1} f_0, \quad h_{k+1} = A^{-1}\left(\frac{dh_k}{dt} - f_{k+1} - \sum_{i=0}^{k} B_i h_{k-i}\right) \tag{2.3}$$

Solution $x(t) = h_0(t) + \varepsilon h_1(t) + \ldots + \varepsilon^N h_N(t)$ differs [2] from a certain particular solution of (2.1) by a quantity of the order of ε^{N+1}.

2) When the operator A has no bounded inverse, the case becomes much more complicated. Let us assume that the number 0 represents an isolated point in the spectrum of A. Then we can express the whole space E in the form of a simple sum $E = E_1 + E_2$ of two subspaces invariant with respect to the operator A, in such a manner that the spectrum of contraction of A and E_1 lies within the left semiplane, while the spectrum of its contraction on E_2 consists of a single null element. A bounded inverse of A exists however on E_1.

In the case under consideration the homogeneous equation

$$\varepsilon \frac{dx}{dt} = Ax + \varepsilon Bx \qquad \left(B = \sum_{k=0}^{\infty} \varepsilon^k B_k\right) \tag{2.4}$$

will possess both, the rapidly decaying solutions and solutions changing slowly with time. To separate these two types of solutions, an analog of the Krylov-Bogoliubov method is used.

Let us denote by P_1 and P_2 the projection operators acting on the subspaces E_1 and E_2, corresponding to the decomposition $E = E_1 + E_2$. Solutions $x(t)$ appear in the form $x(t) = x_1(t) + x_2(t)$ where $x_1(t)$ is the rapidly decaying and $x_2(t)$ is the slowly changing part of the solution.

Functions are constructed according to the formulas

$$x_i(t) = Y_i(t) U_i(t) P_i x_0 \tag{2.5}$$

where U_i is an operator satisfying

$$\varepsilon \frac{dU_i}{dt} = AP_i U_i + \varepsilon S_i U_i, \; U_i(0) = P_i \tag{2.6}$$

Insertion of (2.5) into (2.4) with (2.6) taken into account yields the following equation for Y_i:

$$\varepsilon \frac{dY_i}{dt} P_i = AY_i P_i - Y_i AP_i - \varepsilon Y_i S_i P_i + \varepsilon BY_i P_i \tag{2.7}$$

Operators S_i and Y_i are now sought in the form of series

$$S_i = \sum_{k=0}^{\infty} \varepsilon^k S_i^k, \qquad Y_i = P_i + \sum_{k=1}^{\infty} \varepsilon^k Y_i^k \tag{2.8}$$

Insertion of (2.8) into (2.7) yields the following system of equations defining the coef ficients of expansions:

$$\begin{aligned} \frac{dY_i^k}{dt} P_i = AY_i^{k+1} P_i - Y_i^{k+1} AP_i - P_i S_i^k P_i - \\ - \sum_{j=1}^{k} Y_i^j S_i^{k-j} P_i + \sum_{j=0}^{k} B_{k-j} Y_i^j P_i \end{aligned} \tag{2.9}$$

Let us assume that the operators $Y_i^1, \ldots, Y_i^k$ and $S_i^0, \ldots, S_i^{k-1}$ are already determined in such a manner that

$$Y_i^j = (I - P_i) Y_i^j P_i \quad (j = 1, \ldots, k), \qquad S_i^j = P_i S_i^j P_i \quad (j = 1, \ldots, k-1)$$

and let us find from (2.9) the operators Y_i^{k+1} and S_i^k satisfying the relations

$$Y_i^{k+1} = (I - P_i) Y_i^{k+1} P_i, \qquad S_i^k = P_i S_i^k P_i$$

Operating with P_i on (2.9) we obtain

$$S_i^k = \sum_{j=0}^{k} P_i B_{k-j} Y_i^j P_i \tag{2.10}$$

while the operator $I - P_i$ acting on (2.9) yields

$$\begin{aligned} (I - P_i) AY_i^{k+1} - Y_i^{k+1} AP_i = \frac{dY_i^k}{dt} P_i + \\ + \sum_{j=1}^{k} Y_i^j S_i^{k-j} P_i - \sum_{j=0}^{k} (I - P_i) B_{k-j} Y_i^j P_i \end{aligned} \tag{2.11}$$

By the general theory (see e. g. [2], ch. 4, Lemma 3. 1) the equation obtained has a solution. It can easily be seen that when the operators B_k are constant, Y_i^k are also independent of t.

Thus, the problem of obtaining the Nth approximation to the function $x_i(t)$ is reduced to consecutive solving the operator equations of the form (2. 10) and (2. 11), and consequently to solution of the differential equation

$$\varepsilon \frac{dU_i^N}{dt} = AP_iU_i^N + \varepsilon \sum_{j=0}^{N-1} \varepsilon^j S_i^j U_i^N, \qquad U_i^N(0) = P_i$$

We then have

$$x_i^N(t) = \sum_{j=0}^{N} \varepsilon^j Y_i^j U_i^N P_i x_0 \tag{2.12}$$

This solution does not generally satisfy the initial condition $x_i(0) = P_i x_0$. In fact

$$x_i^N(0) = P_i x_0 + \sum_{j=1}^{N} \varepsilon^j Y_i^j(0) P_i x_0$$

We note that the discrepancy in the initial condition belongs to a subspace complementary to E_i. A method of consecutive elimination of this discrepancy is given in [2] and we apply it below to a particular case.

We seek the particular solutions of the inhomogeneous equation in the form

$$x^*(t) = Y_2 v_2(t) + h(t)$$

where $h(t)$ is defined from E_1 and $v_2(t)$ is a solution of the equation

$$\varepsilon \frac{dv_2}{dt} = AP_2 v_2 + \varepsilon S_2 v_2 + g$$

where g is an auxilliary function defined in E_2.

The requirement that Y_2 and S_2 again satisfy (2. 9) with $i = 2$, yields the following equation for h

$$\varepsilon \frac{dh}{dt} = Ah + \varepsilon Bh + f - Y_2 g$$

Assuming that

$$f = f_0 + f_1\varepsilon + f_2\varepsilon^2 + \ldots, \quad B = B_0 + B_1\varepsilon + B_2\varepsilon^2 + \ldots$$

we seek the functions h and g in the form of expansions

$$h(t) = h_0(t) + \varepsilon h_1(t) + \varepsilon^2 h_2(t) + \ldots$$
$$g(t) = g_0(t) + \varepsilon g_1(t) + \varepsilon^2 g_2(t) + \ldots$$

whose coefficients are defined by

$$Ah_0 = P_2 g_0 - f_0$$

$$\frac{dh_k}{dt} = Ah_{k+1} + \sum_{j=0}^{k} B_{k-j} h_j + f_{k+1} - \sum_{j=0}^{k} Y_2^{k-j+1} g_j - P_2 g_{k+1}$$

Applying the operators P_1 and $P_2 = I - P_1$ to these equations and taking into account the fact that $P_2 h_j = 0$ and $P_1 g_j = 0$, we find

$$g_0 = P_2 f_0, \qquad g_{k+1} = P_2 \sum_{j=0}^{k} B_{k-j} h_j + P_2 f_{k+1} \tag{2.13}$$

$$h_0 = -A_1^{-1} P_1 f_0, \quad h_{k+1} = A_1^{-1} \left\{ \frac{dh_k}{dt} - P_1 \sum_{j=0}^{k} B_{k-j} h_j - P_1 f_{k+1} - P_1 \sum_{j=0}^{k} Y_2^{k-j+1} g_j \right\}$$

where A_1 denotes the contraction of the operator A in E_1.

Thus, to obtain the Nth approximation to some particular solution of (2.1), we must find the functions h_j and g_j from (2.13) and consequently solve the equation

$$\varepsilon \frac{dv_2^N}{dt} = AP_2v_2^N + \varepsilon \sum_{k=0}^{N-1} \varepsilon^k S_2^k v_2^N + \sum_{k=0}^{N} \varepsilon^k g_k \tag{2.14}$$

in the subspace E_2 with an arbitrary initial condition (e.g. $v_2^N(0) = 0$), whereupon the formula

$$x^{*N}(t) = \sum_{j=0}^{N} \varepsilon^j Y_2^j v_2^N(t) + \sum_{k=0}^{N} \varepsilon^k h_k \tag{2.15}$$

gives the required Nth approximation. We note that all the terms on the right side belong to E_1 except $P_2 v_2^N(t)$.

The sum of approximate solutions (2.12), (2.15) obtained, satisfy Eq. (2.1) with the accuracy up to the terms of order of ε^{N+1}. As already indicated in [2], this implies that the approximate solution differs from some actual solution by a magnitude of the order of ε^{N-1}, consequently the only reliable terms in (2.12) and (2.15) will be those containing ε raised to a power not greater than $N - 2$.

3. Motion of a fluid completely filling the cavity. If a fluid fills the cavity completely, then the system of equations (1.1)–(1.4) becomes simpler as conditions (1.3) no longer apply. It was shown in [5] that the resulting problem can be treated as the Cauchy's problem for the following differential equation:

$$\frac{d\mathbf{u}}{dt} = -\nu A\mathbf{u} + P\left(r \times \frac{d\boldsymbol{\omega}}{dt}\right), \qquad \mathbf{u}(0) = \mathbf{u}_0 \tag{3.1}$$

in the Hilbert space N, i.e. as the closure in $L_2(\Omega)$ of the set of all smooth selenoidal vector fields satisfying the condition $|\mathbf{u}_n|_{\Gamma_1} = 0$. Here P is an orthogonal projection operator from L_2 onto N and A is a positive definite self conjugate operator in N. In the equation

$$\varepsilon \frac{d\mathbf{u}}{dt} = -A\mathbf{u} + \varepsilon P\left(\mathbf{r} \times \frac{d\boldsymbol{\omega}}{dt}\right) \qquad (\varepsilon = \nu^{-1}) \tag{3.2}$$

obtained from (3.1), the operator A has a bounded inverse and conditions of the simple case (1) hold. Consequently, by (2.2) and (2.3) the approximate solution of (3.2) has the form

$$\mathbf{u}^N = \sum_{k=0}^{N} \varepsilon^k \mathbf{h}_k(t), \qquad \mathbf{h}_0 = 0, \quad \mathbf{h}_1 = A^{-1}P\left(\mathbf{r} \times \frac{d\boldsymbol{\omega}}{dt}\right)$$

$$\mathbf{h}_N = \left(-A^{-1}\frac{d}{dt}\right)^{N-1} A^{-1}P\left(\mathbf{r} \times \frac{d\boldsymbol{\omega}}{dt}\right) = -\left(-A^{-1}\frac{d}{dt}\right)^{N} P(\mathbf{r} \times \boldsymbol{\omega})$$

Limiting ourselves to the first approximation we have

$$\mathbf{u}^1 = \varepsilon A^{-1}P\left(\mathbf{r} \times \frac{d\boldsymbol{\omega}}{dt}\right) \tag{3.3}$$

Determination of the operator $\varepsilon A^{-1}P$ demands solution of the following problem:

$$\nu\Delta\mathbf{u} = -\nabla s + \left[\frac{d\boldsymbol{\omega}}{dt} \times r\right], \qquad \operatorname{div}\mathbf{u} = 0, \qquad \mathbf{u}|_{\Gamma_1} = 0 \tag{3.4}$$

A solution in the form of (3.3) was obtained in [3] by Chernous'ko, who also showed that the solution of (3.4) can be written in the form of a sum of the "generalized Zhukovskii potentials".

4. Motion of fluid partially filling the cavity. In this case the equations of the problem can also be written in the operator form [1, 6 and 7]

$$\frac{d\mathbf{u}}{dt} + \nu A\mathbf{u} + \Pi\left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}\right) = 0, \quad \nu\frac{d\mathbf{w}}{dt} + gT\Gamma\mathbf{u} = 0, \qquad \mathbf{u} = \mathbf{s} + \mathbf{w} \tag{4.1}$$

where $\mathbf{u}$, $\mathbf{s}$ and $\mathbf{w}$ are functions defined on the space $W_2^{0'}(\Omega)$, the latter being the closure in the S. L. Sobolev space $W_2^1(\Omega)$ of the set of solenoidal vector fields, becoming zero near the Γ_1 part of the boundary. We shall describe the operators A, Π, T and Γ later, now only remarking that the operator A is again positive definite and selfconjugate. After the substitution

$$\mathbf{u} = A^{-1/2}\xi, \quad \mathbf{s} = A^{-1/2}\eta, \quad \mathbf{w} = A^{-1/2}\zeta, \quad \varepsilon = \nu^{-1}, \quad \mathbf{X} = \begin{pmatrix}\eta \\ \zeta\end{pmatrix} \tag{4.2}$$

we can write (4.1) in the form analogous to (2.1)

$$\varepsilon\frac{d\mathbf{X}}{dt} = A_0\mathbf{X} + \varepsilon^2 B_1\mathbf{X} + \varepsilon f_1, \quad A_0 = \begin{Vmatrix} -A & 0 \\ 0 & 0\end{Vmatrix}, \quad B_1 = g\begin{Vmatrix} Q & Q \\ -Q & -Q\end{Vmatrix}, \quad f_1 = \begin{Vmatrix}\varphi \\ 0\end{Vmatrix}$$

$$\varphi = A^{1/2}\Pi\left(\mathbf{r} \times \frac{d\boldsymbol{\omega}}{dt}\right), \qquad Q = A^{1/2}T\Gamma A^{-1/2} \tag{4.3}$$

Whole of the space E of vectors $\mathbf{X}$ can naturally be expressed as a simple sum of two subspaces E_1 and E_2 composed, respectively, of vectors of the form $\{\eta, 0\}$ and $\{0, \zeta\}$. In E_1 the operator A_0 is negative definite and has a bounded inverse

$$A_0^{-1}\mathbf{X} = \begin{pmatrix}-A^{-1}\eta \\ 0\end{pmatrix} \qquad (\mathbf{X} \in E_1)$$

In E_2 the operator A_0 is identically equal to zero, therefore the projection operators P_1 and P_2 have the form

$$P_1 = \begin{Vmatrix} I & 0 \\ 0 & 0\end{Vmatrix}, \qquad P_2 = \begin{Vmatrix} 0 & 0 \\ 0 & I\end{Vmatrix}$$

Following the scheme given in Sect. 2, let us limit ourselves to the third approximations to the solution of (4.3). From (2.10) and (2.11) we find

$$Y_1^{(0)} = P_1,\ S_1^{(0)} = 0;\ Y_1^{(1)} = 0,\ S_1^{(1)} = P_1B_1P_1;\ Y_1^{(2)} = P_2B_1P_1,\ S_1^{(2)} = 0;\ Y_1^{(3)} = 0$$

Expressing the operators in matrix form we obtain

$$Y_1^{(0)} = P_1, \quad S_1^{(0)} = 0; \qquad Y_1^{(1)} = 0, \quad S_1^{(1)} = \begin{Vmatrix} gQ & 0 \\ 0 & 0\end{Vmatrix}$$

$$Y_1^{(2)} = \begin{Vmatrix} 0 & 0 \\ gQA^{-1} & 0\end{Vmatrix}, \quad S_1^{(2)} = 0; \qquad Y_1^{(3)} = 0$$

In a similar manner we obtain

$$Y_2^{(0)} = P_2, \quad S_2^{(0)} = 0; \qquad Y_2^{(1)} = 0, \quad S_2^{(1)} = \begin{Vmatrix} 0 & 0 \\ 0 & -gQ\end{Vmatrix}$$

$$Y_2^{(2)} = \begin{Vmatrix} 0 & gA^{-1}Q \\ 0 & 0\end{Vmatrix}, \quad S_2^{(2)} = 0; \qquad Y_2^{(3)} = 0$$

Differential equations (2.6) for the operators $U_1^{(3)}$ and $U_2^{(3)}$ now become

$$\varepsilon\frac{dU_i^{(3)}}{dt} = A_0P_iU_i^{(3)} + \varepsilon^2S_i^{(1)}U_i^{(3)}, \qquad U_i^{(3)}(0) = P_i$$

or in the subspaces E_1 and E_2,

$$\varepsilon\frac{dU_1^{(3)}}{dt} = -AU_1^{(3)} + \varepsilon^2gQU_1^{(3)}, \qquad U_1^{(3)}(0) = I$$

$$\varepsilon\frac{dU_2^{(3)}}{dt} = -\varepsilon^{(2)}gQU_2^{(3)}, \qquad U_2^{(3)}(0) = I$$

Thus, we have split the basic differential equation into two equations, first of which has rapidly decaying solutions while the other has solutions varying slowly with time.

For the third approximation to the solution of the homogeneous equation corresponding to (4.3) we obtain

$$\mathbf{X}^{(3)} = \left\| \begin{matrix} U_1^{(3)}\eta_0 + \varepsilon^2 g A^{-1} Q U_2^{(3)}\zeta_0 \\ U_2^{(3)}\zeta_0 + \varepsilon^2 g Q A^{-1} U_1^{(3)}\eta_0 \end{matrix} \right\| \tag{4.4}$$

For a particular solution of the inhomogeneous equation, (2.13) and (2.14) yield

$$\mathbf{h}_0 = 0, \quad \mathbf{g}_0 = 0; \quad \mathbf{h}_1 = A^{-1}\mathbf{f}_1, \quad \mathbf{g}_1 = 0; \quad \mathbf{h}_2 = -A^{-2}\frac{d\mathbf{f}_1}{dt}, \quad \mathbf{g}_2 = 0$$

$$\mathbf{h}_3 = \left\| \begin{matrix} A^{-3}\dfrac{d\mathbf{f}_1}{dt} \\ 0 \end{matrix} \right\| + \mathbf{g} \left\| \begin{matrix} A^{-1} Q A^{-1}\mathbf{f}_1 \\ 0 \end{matrix} \right\|, \qquad \mathbf{g}_3 = \left\| \begin{matrix} 0 \\ -g Q A^{-1}\mathbf{f}_1 \end{matrix} \right\|$$

and we then have

$$\mathbf{X}^{*(3)} = \left\| \begin{matrix} \mathbf{v}_1^{(3)} \\ \mathbf{v}_2^{(3)} \end{matrix} \right\|, \qquad \mathbf{v}_1^{(3)} = \varepsilon^2 g A^{-1} Q \mathbf{v}_2^{(3)} + \varepsilon A^{-1/2}\Pi\left(\mathbf{r} \times \frac{d\boldsymbol{\omega}}{dt}\right) -$$

$$- \varepsilon^2 A^{-3/2}\Pi\left(\mathbf{r} \times \frac{d^2\omega}{dt^2}\right) + \varepsilon^3 A^{-5/2}\Pi\left(r \times \frac{d^3\boldsymbol{\omega}}{dt^3}\right) + \varepsilon^3 g A^{-1} Q A^{-1}\Pi\left(\mathbf{r} \times \frac{d\boldsymbol{\omega}}{dt}\right)$$

where $v_2^{(3)}$ is a solution of the following differential equation:

$$\varepsilon \frac{d\mathbf{v}_2^{(3)}}{dt} = -\varepsilon^2 \mathbf{g} Q \mathbf{v}_2^{(3)} - \varepsilon^3 \mathbf{g} Q A^{-1/2}\Pi\left(\mathbf{r} \times \frac{d\omega}{dt}\right), \qquad \mathbf{v}_2^{(3)}(0) = 0 \tag{4.5}$$

Operator Q appearing in this equation is a nonnegative self conjugate operator in the space $W_2^{0\prime}$ (see e.g. [6 and 7]). It can easily be seen that it becomes positive in the subspace E_2.

The sum $\mathbf{X}^3 + \mathbf{X}^{*3}$ gives the third approximation to some solution of the inhomogeneous equation, but, as we have already remarked, its only reliable terms will be those containing ε in the degree not greater than first. Thus, the approximate solution differing from the exact one in terms of order of ε^2, is

$$\left\| \begin{matrix} U_1^{(3)}\eta_0 + \varepsilon A^{-1/2}\Pi\left(\mathbf{r} \times \dfrac{d\omega}{dt}\right) \\ U_2^{(3)}\zeta_0 \end{matrix} \right\|$$

in obtaining which we have assumed that the solution of the problem (4.5) is of order of ε^2.

The solution just obtained does not satisfy the given initial conditions. Indeed, when $t = 0$, its components are, respectively,

$$\left(\eta_0 + \varepsilon A^{-1/2}\Pi\left(\mathbf{r} \times \left(\frac{d\boldsymbol{\omega}}{dt}\right)_0\right)\right), 0$$

It follows therefore that such approximate solution should be deduced from it, which would satisfy the homogeneous equation (4.3) with the accuracy of up to the terms of order of ε^2 and which would have the initial value

$$\varepsilon \left\| \begin{matrix} A^{-1/2}\Pi\left(r \times \left(\dfrac{d\omega}{dt}\right)_0\right) \\ 0 \end{matrix} \right\|$$

Such a solution can be constructed from (4.4) by replacing $U_i^{(3)}$ with $U_i^{(2)}$ which in this case coincide (since $S_i^{(2)} = 0$). Retaining again only the reliable terms, we obtain the final formula for the first approximation to the solution of the problem under consideration

$$\mathbf{X} = \left\| \begin{matrix} U_1^{(3)}(\eta_0 - \varepsilon A^{-1/2}\Pi\left(\mathbf{r}\times\left(\frac{d\boldsymbol{\omega}}{dt}\right)_0\right) + \varepsilon A^{-1/2}\Pi\left(\mathbf{r}\times\frac{d\boldsymbol{\omega}}{dt}\right) \\ U_2^{(3)}\zeta_0 \end{matrix} \right\|$$

Performing the substitution (4.2) and taking into account the fact that $\mathbf{u} = \mathbf{s} + \mathbf{w}$, we find

$$u = A^{-1/2}U_1^{(3)}\left(A^{1/2}\mathbf{s}_0 - \varepsilon A^{-1/2}\Pi\left(\mathbf{r}\times\left(\frac{d\boldsymbol{\omega}}{dt}\right)_0\right)\right) + \varepsilon A^{-1}\Pi\left(\mathbf{r}\times\frac{d\boldsymbol{\omega}}{dt}\right) + A^{-1/2}U_2^{(3)}A^{1/2}w_0 \quad (4.6)$$

First term of this formula describes the rapidly decaying motion, second term the forced motion and the third term – the slowly decaying motion. Discarding the rapidly decaying terms we obtain

$$\mathbf{u} = A^{-1/2}U_2^{(3)}A^{1/2}\mathbf{w}_0 + \varepsilon A^{-1}\Pi\left(\mathbf{r}\times\frac{d\boldsymbol{\omega}}{dt}\right) \quad (4.7)$$

We shall now describe the procedure of obtaining the first approximation [1, 6 and 8].

F o r c e d m o t i o n. We solve the following boundary value problems

$$-\nu\Delta\mathbf{s}_i + \nabla p_i = \mathbf{r}\times\mathbf{e}_i, \quad \operatorname{div}\mathbf{s}_i = 0, \quad \mathbf{s}_i = 0 \text{ on } \Gamma_1$$

$$\frac{\partial s_{iy}}{\partial z} + \frac{\partial s_{iz}}{\partial y} = 0, \ \frac{\partial s_{ix}}{\partial z} + \frac{\partial s_{iz}}{\partial x} = 0, \quad -p_i + 2\nu\frac{\partial s_{iz}}{\partial z} = 0 \text{ on } \Gamma_0$$

where $\mathbf{e}_i$ denote unit vectors along the axes.

Then the relative velocity of the forced motion will be equal to

$$\mathbf{u}_2 = \varepsilon_1\mathbf{s}_2 + \varepsilon_2\mathbf{s}_2 + \varepsilon_3\mathbf{s}_3$$

where ε_i are the projections of the angular acceleration of the body on the axes of the moving coordinate system.

If the problem calls for the detrmination of pressures appearing in the fluid, then we must solve the boundary value problems for the Laplace's equation

$$\Delta\varphi_i = 0; \qquad \varphi_i = 0 \text{ on } \Gamma_1, \qquad \frac{\partial\varphi_i}{\partial n} = (r\times e_i) \quad \text{on } \Gamma_0$$

Then

$$\Pi\left(\mathbf{r}\times\frac{d\boldsymbol{\omega}}{dt}\right) = \sum_{i=1}^{3}\varepsilon_i\ (\mathbf{r}\times\mathbf{e}_i - \operatorname{grad}\varphi_i)$$

Pressure p is given by

$$p = \rho gz + \rho[\varepsilon_1(p_1 - \varphi_1) + \varepsilon_2(p_2 - \varphi_2) + \varepsilon_3(p_3 - \varphi_3)]$$

S l o w l y d e c a y i n g m o t i o n. Operator function $V = A^{-1/2}U_2^3A^{1/2}$ is a solution of

$$dV/dt = -\varepsilon gT\Gamma V, \qquad V(0) = I$$

Using the classical terminology we can now formulate the rule for obtaining a solution. Solution of the following problem is required:

$$-\nu\Delta\mathbf{w} + \nabla p = 0, \qquad \operatorname{div}\mathbf{w} = 0, \qquad \mathbf{w} = 0 \quad \text{on } \Gamma_1$$

$$\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x} = 0, \quad \frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y} = 0; \quad \frac{\partial}{\partial t}\left(-p + 2\nu\frac{\partial w_z}{\partial z}\right) = -gw_z \quad \text{on } \Gamma_0$$

Then the relative velocity of a slowly decaying motion is:

$$u_3 = w_0 + \int_0^t w\,dt$$

F r e e o s c i l l a t i o n s. When considering the problem of free oscillations of a strongly viscous fluid in a motionless vessel, we can utilize the expression (4.7) with $\boldsymbol{\omega} = 0$ to obtain normal oscillations proportional to $e^{-\lambda t}$. In the case of slow oscillations, the quantity λ is the eigenvalue of the self conjugate problem

$$-\nu\Delta\mathbf{w} + \nabla p = 0, \quad \operatorname{div}\mathbf{w} = 0; \quad \mathbf{w} = 0 \text{ on } \Gamma_1$$

$$\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x} = 0, \qquad \frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y} = 0 \qquad \lambda\left(-p + 2\nu\frac{\partial w_z}{\partial z}\right) = gw_z \text{ on } \Gamma_0$$

To determine the rate of decay of the rapid motions we must consider the first term of (4.6). The operator function $S = A^{-1/2}U_1^{(3)}A^{1/2}$ satisfies the equation

$$\varepsilon\frac{dS}{dt} = -AS + \varepsilon^2 gT\Gamma S, \qquad S(0) = I$$

which can be replaced by another, simpler equation

$$\varepsilon \frac{dS}{dt} = -AS$$

with the accuracy of up to the terms of order of ε^2.

For normal oscillations the problem is $\nu A s = \lambda s$

and using the classical formulation we obtain the following self conjugate eigenvalue problem

$$-\nu\Delta \mathbf{s} + \Delta p = \lambda \mathbf{s}, \quad \operatorname{div} \mathbf{s} = 0, \quad s = 0 \quad \text{on } \Gamma_1$$

$$\frac{\partial s_x}{\partial z} + \frac{\partial s_z}{\partial x} = 0, \quad \frac{\partial s_y}{\partial z} + \frac{\partial s_z}{\partial y} = 0; \quad -p + 2\nu \frac{\partial s_z}{\partial z} = 0 \text{ on } \Gamma_0$$

5. Combined motion of the body and fluid. Equation of angular momentum for the system "body + fluid" has the form [1 and 9]

$$I \frac{d\boldsymbol{\omega}}{dt} + \rho \int_{\Omega} \left(\mathbf{r} \times \frac{d\mathbf{u}}{dt}\right) d\Omega + \mathbf{M} = 0 \qquad (5.1)$$

$$\mathbf{M} = mga\,(\delta_1 \mathbf{e}_1 + \delta_2 \mathbf{e}_2) + \rho g\,(k_1 \times \int_{\Gamma_0} r f d\Gamma_0) \qquad (5.2)$$

where m is the mass of the system, a is the distance between the center of mass of the system and the fixed point, δ_i are the components of the angular displacement vector in the moving coordinate system, k_1 is the unit vector along the moving Oz axis and $z = f(x, y, t)$ is the equation of the free surface in the moving coordinate system.

When the velocity of motion is known, the function f is given by

$$f(x, y, t) = \int_0^t u_z\, d\tau + f(x, y, 0) \qquad (5.3)$$

Inserting the expression (4.6) for the velocity $\mathbf{u}$ into (5.1)–(5.3), we obtain a third order differential equation defining the components of the angular displacement vector of the body in the first approximation

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